

CODIMENSION ZERO LAMINATIONS ARE INVERSE LIMITS

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ABSTRACT. The aim of the paper is to investigate the relation between inverse limit of branched manifolds and codimension zero laminations. We give necessary and sufficient conditions for such an inverse limit to be a lamination. We also show that codimension zero laminations are inverse limits of branched manifolds.

The inverse limit structure allows us to show that equicontinuous codimension zero laminations preserves a distance function on transversals.

1. INTRODUCTION

Consider the circle $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and the cover of degree 2 of it $p_2(z) = z^2$. Define the inverse limit

$$\mathbf{S}_2 = \varprojlim (\mathbb{S}^1, p_2) = \{ (z_k) \in \prod_{k \geq 0} \mathbb{S}^1 \mid z_k^2 = z_{k-1} \}.$$

This space has a natural foliated structure given by the flow $\Phi_t(z_k) = (e^{2\pi i t/2^k} z_k)$. The set $X = \{ (z_k) \in \mathbf{S}_2 \mid z_0 = 1 \}$ is a complete transversal for the flow homeomorphic to the Cantor set. This space is called *solenoid*. This construction can be generalized replacing \mathbb{S}^1 and p_2 by a sequence of compact p -manifolds and submersions between them. The spaces obtained this way are compact laminations with 0 dimensional transversals.

This construction appears naturally in the study of dynamical systems. In [17, 18] R.F. Williams proves that an expanding attractor of a diffeomorphism of a manifold is homeomorphic to the inverse limit

$$S \xleftarrow{f} S \xleftarrow{f} S \xleftarrow{f} \dots$$

where f is a surjective immersion of a branched manifold S on itself. A branched manifold is, roughly speaking, a CW-complex with tangent space at each point.

After their introduction by Williams, branched manifolds and their limits have been extensively used in the study of dynamical systems and foliations. For example W. Thurston uses *train tracks* (1-dimensional branched manifolds) in geodesic laminations on hyperbolic surfaces [15]. Later, J. Anderson and I. Putnam show [2] that substitution tiling spaces are inverse limits of a CW-complex as in the case of Williams. J. Bellissard, R. Benedetti and J-M. Gambaudo [3], F. Gähler [9] and L. Sadun [14] independently extended this result to any tiling, showing that they are inverse limits of branched manifolds. In this case, the projective system has different branched surfaces

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at each level. With a similar scheme as in [3] R. Benedetti and J.M. Gambaudo has extended in [4] the previous result to \mathbb{G} -solenoids (free actions of a Lie group \mathbb{G} with transverse Cantor structure).

F. Alcalde Cuesta, M. Macho Stadler and the author prove in [1] that any compact without holonomy minimal lamination of codimension 0 is an inverse limit, generalizing all previous results.

In this article, we thoroughly explore the relation between inverse limits of branched manifolds and laminations with zero dimensional transverse structure. Let us start considering the following example. Take the eight figure

$$K = \mathbb{S}^1 \wedge_1 \mathbb{S}^1 = \mathbb{S}^1 \times \{1, 2\} / (1, 1) \sim (1, 2),$$

that is, the two copies of the circle glued by the 1. For each copy of \mathbb{S}^1 we have the degree two covering p_2 , so we can define $P_2([z, i]) = [z^2, i]$, where $[z, i] \in K$. Let X be the inverse limit $\varprojlim (K, P_2)$. It is easy to see that X is homeomorphic to $\mathbf{S}_2 \wedge_{(1)} \mathbf{S}_2$, i.e., two copies of the solenoid glued by the sequence $(1) \in \mathbf{S}_2$. It is clear that it is not a lamination. The problem is that P_2 does not *iron out* the branching in each step, collapsing the branches at branching points to one single disk. The kind of maps doing that are called *flattening* [3].

We have three main theorems in the paper. Firstly, we show that this is a necessary and sufficient condition on an inverse systems to obtain a lamination as it limit:

Theorem 4.3. *Fix a projective system (B_k, f_k) where B_k are branched n -manifolds and f_k cellular maps, both of class C^r . The inverse limit B_∞ of the system is a codimension zero lamination of dimension n and class C^r if and only if the systems is flattening.*

Secondly, thinking in laminations as tiling spaces [1] we can adapt the constructions for tilings [9, 14] to obtain a result in the other direction: from laminations to systems of branched manifolds. This theorem extends [1] to any lamination of codimension zero:

Theorem 5.8. *Any codimension zero lamination (M, \mathcal{L}) is homeomorphic to an inverse limit $\varprojlim (S_k, f_k)$ of branched manifolds S_k and cellular maps $f_k : S_k \rightarrow S_{k-1}$.*

Finally, the inverse limit structure can give information of the dynamics of the space. M.C. McCord [13] and recently A. Clark and S. Hurder [7] study an important class of solenoidal spaces, those given by real manifolds and regular covering maps as bounding maps. With this structure we can conclude that:

Theorem 6.3. *An equicontinuous lamination of codimension zero preserves a transverse metric.*

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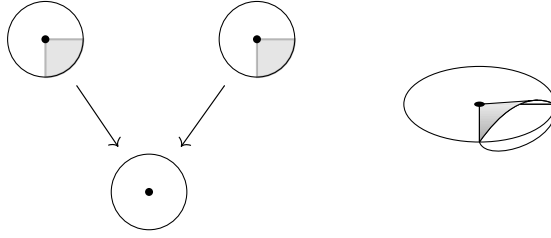


FIGURE 1. A tree and the local model.

2. BRANCHED MANIFOLDS

Let \mathbb{D}^n denote the closed n -dimensional unit disk. A *sector* is the, eventually empty, interior of the intersection of a (finite) family of half-spaces through the origin. Fix a finite family of sectors \mathcal{S} , and a finite directed tree T with a map $s : VT \rightarrow \mathcal{S}$ where VT is the vertex set of T .

Now define a *local branched model* U_T as the quotient set of $\mathbb{D}^n \times VT$ by the relation generated by

$$(x, v) \sim (x, v') \iff \text{the edge } v \rightarrow v' \text{ exists in } T \text{ and } x \in \mathbb{D}^n \setminus s(v).$$

The quotient $D_v \subseteq U_T$ of each set $\mathbb{D}^n \times \{v\}$ is called a *smooth disks*. There is a natural map $\Pi_T : U_T \rightarrow \mathbb{D}^n$ given by the quotient of the first coordinate projection $pr_1 : (x, v) \mapsto x$, which is a homeomorphism restricted to each smooth disk.

Definition 2.1. A *branched manifold of class C^r of dimension n* is a Polish space S endowed with an atlas of closed disks $\{U_i\}$ homeomorphic to local branched models of dimension n such that there is a cocycle of C^r -diffeomorphisms $\{\alpha_{ij}\}$ between open sets of \mathbb{D}^n fulfilling $\Pi_i \circ \alpha_{ij} = \Pi_j$, where Π_i denotes the natural map of the local models.

Remark 2.2. Definition 2.1 is not the classical one [18]. In our setting the branching behavior is quite simple as we have locally finite branching. This is not true with the classical definition.

Following [18], there is a natural notion of differentiable map: a map $f : S \rightarrow S'$ between two branched manifolds of class at least $r \geq 1$ is of class C^r if the local map

$$f_{T,v}^{T'} : \Pi_T(D_v) \xrightarrow{(\Pi|_{D_v})^{-1}} D_v \xrightarrow{f} U_{T'} \xrightarrow{\Pi_{T'}} \mathbb{D}^n$$

is of class C^r and the germ of $f_{T,v}^{T'}$ at any point does not depends on T' .

Of course, there is the notion of tangent fibre bundle for branched manifolds. For each smooth disk D_v of a local model U_T we have the induced tangent bundle $(\Pi_T|_{D_v})^* T\mathbb{R}^n \cong D_v \times \mathbb{R}^n$. The tangent bundle of U_T is just the *gluing* of those, which is naturally isomorphic to $U_T \times \mathbb{R}^n$. Finally, as the change of coordinates are diffeomorphisms between smooth disks there is a well defined tangent bundle. Obviously, one can define higher bundles provided the differentiability class is high enough. As in the classical setting differentiable maps induce bundle maps. Then the usual definitions of *submersion* and *immersion* make sense. Anyway, we should remark that a

submersion does not need to be locally surjective and an immersion does not need to be locally injective.

3. CODIMENSION ZERO LAMINATIONS

Let M be a locally compact Polish space. A *foliated atlas of class C^r* is an atlas $\mathcal{A} = \{\varphi_i : U_i \rightarrow \mathbb{R}^n \times X_i\}$ of M such that the changes of coordinates can be written as

$$(1) \quad \varphi_i \circ \varphi_j^{-1}(x, y) = (\varphi_{ij}^y(x), \sigma_{ij}(y)),$$

where σ_{ij} is a homeomorphism and φ_{ij}^y is a C^r -diffeomorphism depending continuously on y in the C^r -topology. Two foliated atlases of class C^r are *equivalent* if the union of them is still a foliated atlas. A *lamination* of class C^r is a base space M endowed with an equivalence class \mathcal{L} of foliated atlas. The open sets U_i are usually called *flow boxes* and the level sets $\varphi_i^{-1}(P_i \times \{y\})$ and $\varphi_i^{-1}(\{x\} \times X_i)$ are the *plaques* and *local transversals* of \mathcal{A} respectively. A *transversal* of \mathcal{L} is just a union of local transversals of a compatible atlas. The *leaves* L of \mathcal{L} are the smallest path connected sets such that if L meets a plaque P , then $P \subset L$. The plaques define a n -manifold structure of class C^r on each leaf. It is usual to identify the lamination \mathcal{L} with the partition formed by the leaves, so it is common to write $L \in \mathcal{L}$ for a leaf L of \mathcal{L} . Finally, a *complete transversal* is a transversal meeting all leaves. The space $T = \bigsqcup_i X_i$ is the *complete transversal associated to the atlas \mathcal{A}* . An atlas \mathcal{A} is *good* if it is finite and

- (1) the flow boxes U_i are relatively compact;
- (2) if $U_i \cap U_j \neq \emptyset$, there is a flow box U_{ij} containing $\overline{U_i \cap U_j}$ and therefore each plaque of U_i intersects at most one plaque of U_j .

The codimension of the lamination \mathcal{L} , denoted by $\text{codim } \mathcal{L}$, is the dimension of the complete transversal associated to any atlas. Then we talk about codimension zero lamination if the transversal is a locally compact Polish 0-dimensional space.

Remark 3.1. Given a codimension zero laminations, the leaves of the underlying laminations are the path connected components, as local transversals are totally disconnected. Hence the foliated structure is intrinsic to the topological space M .

Example 3.2. Consider the Cantor set $C = \{0, 1\}^{\mathbb{Z}}$ and the shift map $\sigma((x_i))_k = x_{k+1}$, for $(x_i) \in C$. We can consider then the suspension of this homeomorphism: Consider the action of \mathbb{Z} on $\mathbb{R} \times C$ given by

$$(\lambda, x) + \ell = (\lambda - \ell, \sigma^\ell(x)),$$

for $\ell \in \mathbb{Z}$. The horizontal foliation on $\mathbb{R} \times C$ given by the leaves $\mathbb{R} \times \{*\}$ is invariant under this action, so it induces a lamination of class C^∞ on the quotient $M = \mathbb{R} \times C / \mathbb{Z}$. For other examples of codimension zero laminations see [10, 5, 11, 12].

Recall that a pseudogroup of transformations on a space T is a family of homeomorphisms from open sets to open sets of T closed under restriction to open sets, composition, inversion and extension. To describe the behavior of the leaves we need the *holonomy pseudogroup*: Fix a good foliated atlas \mathcal{A}

for a lamination \mathcal{L} on M . Fix two meeting flow boxes U_i and U_j and define the corresponding *holonomy transformation* $\sigma_{ij} : D_{ij} \subset X_i \rightarrow D_{ji} \subset X_j$ given by

$$\sigma_{ij}(y_i) = y_j \iff \text{the plaques through } y_i \text{ and } y_j \text{ meet,}$$

where D_{ij} and D_{ji} are the obvious open sets of X_i and X_j respectively. It can be shown σ_{ij} is a homeomorphism. The pseudogroup Γ on T generated by all those maps is the *transverse holonomy pseudogroup associated to \mathcal{A}* .

Given a point $x \in T$ we can consider the group Γ_x of germs of elements of Γ fixing x . Given other element $y \in T$ in the same leaf L , Γ_x and Γ_y are naturally isomorphic. We call this group the *holonomy group* of L and we write Γ_L . In fact, the holonomy group is a quotient of the fundamental group of L : Take a path γ starting and ending at x contained in the corresponding leaf. Then it can be covered by a finite family charts. Making the composition of the corresponding holonomy maps, we obtain a map σ_γ depending on the homotopy class of the path. We have the *holonomy representation* of the fundamental group of L . Details about laminations can be found in [6].

4. FROM SEQUENCES OF BRANCHED MANIFOLDS TO LAMINATIONS

For us, a *projective system* is family of spaces S_k and continuous onto maps $f_k : S_{k+1} \rightarrow S_k$ with $k \in \mathbb{N}$. Given such a system it is possible to define the *projective* or *inverse limit* as

$$S_\infty = \varprojlim (S_k, f_k) = \{ (b_k) \in \prod_{k \geq 0} S_k \mid f_k(b_{k+1}) = b_k \}.$$

For each S_k , there is a natural map $\hat{f}_k : S_\infty \rightarrow S_k$, the restriction of the corresponding coordinate projection. Projective limits has a universal property: if there are maps $g_k : Y \rightarrow S_k$ such that $f_k \circ g_k = g_{k-1}$ for a space Y and all k , then there exists a unique continuous map $g_\infty : Y \rightarrow S_\infty$ such that $\hat{f}_k \circ g_\infty = g_k$.

In general inverse limits might be empty, but if each factor S_k is compact and nonempty, the projective limit will be compact and non empty. Moreover, if the factors are Hausdorff the inverse limit will be too [8].

Given an infinite set $\{a_i\}_{i \in \mathbb{N}} \subset \mathbb{N}$ we define the *telescoping* (associated to the family $\{a_i\}$) of the projective system $\{S_k, f_k : S_k \rightarrow S_{k-1}\}$ as the projective system

$$S_{a_0} \xleftarrow{f_{a_1} \circ \dots \circ f_{a_0+2} \circ f_{a_0+1}} S_{a_1} \xleftarrow{f_{a_2} \circ \dots \circ f_{a_1+2} \circ f_{a_1+1}} S_{a_2} \dots$$

The telescoping is, roughly speaking, a less detailed projective system, as we have dropped some levels, but with this process no information is lost. It is straightforward to show:

Lemma 4.1 (Thm. 2.7 of Apx. 2 [8]). *A projective system and any telescopic contraction of it have the same inverse limit.*

Definition 4.2. A cellular $f : S \rightarrow S'$ map between branched manifolds is *flattening* if for each $b \in S$ there exists a (normal) neighborhood U such that $f(U)$ is a smooth disk of S' . A projective system is *flattening* if there is a telescopic contraction of it with all bond maps flattening.

Theorem 4.3. *Fix a projective system (S_k, f_k) where S_k are branched n -manifolds and f_k cellular maps, both of class C^r . The inverse limit S_∞ of the system is a codimension zero lamination of dimension p and class C^r if and only if the system is flattening.*

Proof. Let us start supposing that the system is flattening, we can assume the maps f_k are flattening immersions. And denote by $f_{k,k'}$ the maps

$$f_{k,k'} = f_k \circ f_{k-1} \circ \cdots \circ f_{k'+1} : S_k \rightarrow S_{k'},$$

where $k > k'$. We will construct a foliated atlas for $S_\infty = \varprojlim (S_k, f_k)$.

Given a thread $(x_k) \in S_\infty$ there are two possibilities: a) $x_{k_0} \in S_{k_0} \setminus \text{Sing } S_{k_0}$ for some $k_0 \in \mathbb{N}$, or b) $x_k \in \text{Sing } S_k$ for all k .

a) As $x_{k_0} \notin \text{Sing } S_{k_0}$ there is a normal neighbourhood D_{k_0} of b_{k_0} diffeomorphic to an Euclidean closed disk. As maps f_k are cellular the space $f_{k,k_0}^{-1}(D_{k_0})$ is a finite disjoint union of copies of D_{k_0} (indeed one for each element of $f_{k,k_0}^{-1}(x_{k_0})$), each of them being again a normal neighbourhood, for each $k > k_0$. Define the set

$$(2) \quad U_{D_{k_0}} = \{ (x'_k) \in S_\infty \mid x'_k \in f_{k,k_0}^{-1}(D_{k_0}) \},$$

which can be also obtained as the inverse limit of the systems $(f_{k,k_0}^{-1}(D_{k_0}), f_k)$. Each $f_{k,k_0}^{-1}(D_{k_0})$ is canonically homeomorphic to $D_{k_0} \times f_{k,k_0}^{-1}(x_{k_0})$. But $D_{k_0} \times f_{k,k_0}^{-1}(x_{k_0})$ with maps $\text{id}_{D_{k_0}} \times f_k$ is an inverse system, and the canonical homeomorphism of each level pass to the limit, therefore $U_{x_{k_0}}$ is homeomorphic to $\varprojlim (D_{k_0} \times f_{k,k_0}^{-1}(x_{k_0}), \text{id}_{D_{k_0}} \times f_k) = D_{k_0} \times \hat{f}_{k_0}^{-1}(x_{k_0})$. Finally, notice that $\hat{f}_{k_0}^{-1}(x_{k_0})$ is a closed and totally disconnected set.

b) In this case, the construction is similar, but now normal neighbourhoods are no longer disks. Given a normal neighbourhood D_0 of x_0 we define the closed set U_{D_0} as in (2). We should use flattening condition to get a product structure. Consider local coordinates for D_0 , that is, a finite disjoint union X_0 of disks, and a map $\varphi_0 : X_0 \rightarrow D_0$. We can assume that each of them is a smooth section for f_0 . Now, for each point of $f_0^{-1}(x_0)$ we can give local coordinates being smooth sections of f_1 . All those local sections form X_1 a disjoint union of disks, and we also have the coordinate map $\varphi_1 : X_1 \rightarrow f_0^{-1}(D_0)$. There is an obvious map g_0 from X_1 to X_0 defined by f_0 . Continuing in this way we get a projective system (X_k, g_k) and a system of maps $\{\varphi_k\}$ to the system $(f_{k,0}^{-1}(D_0), f_k)$. It is straightforward to show that $\varphi_{k-1} \circ g_k = f_k \circ \varphi_k$. Therefore, there is a continuous map $\varphi_\infty : \varprojlim X_k \rightarrow \varprojlim f_{k,0}^{-1}(D_0)$. It is obvious φ_∞ is onto, so it is enough that see that it is injective.

Fix (y_k) and $(y'_k) \in \varprojlim X_k$ such that $\varphi_\infty(y_k) = \varphi_\infty(y'_k)$. Then, for any k $\varphi_k(y_k) = \varphi_k(y'_k)$. But f_k is flattening, therefore y_{k-1} and y'_{k-1} belongs to the same disk in X_k . Thus $y_{k-1} = y'_{k-1}$. By induction we get $(y_k) = (y'_k)$.

The change of coordinates of all those maps are like in (1) as the transversals are totally disconnected.

Now, suppose flattening condition is not fulfilled, even after any telescopic contraction. So there exists a thread $(x_k) \in S_\infty$ such that for each point

$x_k \in S_k$ there are two smooth sections D_k^0 and D_k^1 of f_k such that $f_k : D_{k+1}^i \rightarrow D_k^i$ is a diffeomorphism for $i = 0, 1$. In particular, f_k restricted to the union $D_{k+1}^0 \cup D_{k+1}^1$ is a diffeomorphism of branched manifolds. Therefore the inverse limit is diffeomorphic to $D_{k+1}^0 \cup D_{k+1}^1$ and it is contained in S_∞ . Hence it cannot be a lamination. \square

5. FROM LAMINATIONS TO SEQUENCES OF BRANCHED MANIFOLDS

5.1. Box decompositions.

Definition 5.1. A family of compact flow boxes $\mathcal{B} = \{\varphi_i : B_i \rightarrow D_i \times C_i\}_{i=1}^m$ is said to be a *box decomposition* of (M, \mathcal{L}) if

- (1) the family \mathcal{B} covers M ;
- (2) each transversal C_i is a clopen set of X ;
- (3) if $i \neq j$, the intersection of B_i and B_j agrees with the intersection of the vertical boundaries $\partial_{\text{th}} B_i = \varphi_i(\partial D_i \times C_i)$ and $\partial_{\text{th}} B_j = \varphi_j(\partial D_j \times C_j)$;
- (4) each plaque of B_i meets at most one plaque of B_j ;
- (5) the changes of coordinates are given by:

$$(3) \quad \varphi_i \circ \varphi_j^{-1}(x, y) = (\varphi_{ij}(x), \sigma_{ij}(y)).$$

If the plaques of \mathcal{B} are p -simplexes, each pair of plaques meets in a common face. In this case, we may suppose that the maps φ_{ij} are linear and we will say that \mathcal{B} is a *simplicial box decomposition* (or simply *simplicial decomposition*) of (M, \mathcal{L}) .

Theorem 5.2 ([1]). *Any codimension zero compact C^1 lamination (M, \mathcal{L}) admits a simplicial box decomposition \mathcal{B} .*

The proof of the Theorem is long, but the main points of it are the same of the Classical Triangulation Theorem with the approach inaugurated by A. Weil [16].

5.2. Patterns and boxes. For now on, we fix a simplicial box decomposition $\mathcal{K} = \{\varphi_\Delta : B_\Delta \rightarrow \Delta \times X_\Delta\}$, where Δ runs over a finite set of n -simplexes \mathcal{P} . By analogy, we will call those elements (proto)tiles. Each transversal X_Δ can be identified with the set of centers of the triangles of the corresponding box B_Δ , or in other words, each simplex Δ is pointed at its center. We denote by X the complete transversal associated to \mathcal{K} , that is, the disjoint union $\bigsqcup_\Delta X_\Delta$.

Definition 5.3. A *pattern* P is a finite union (face by face) of copies of elements of \mathcal{P} . If we fix a base point p (chosen from the ones of the tiles forming P) we talk about a *pointed pattern* (P, p) . The base point is dropped from notation if it is clearly understood.

In general, patterns are not subsets of leaves of \mathcal{L} , not even subsets of M . The natural place for patterns is the *holonomy coverings of leaves*, that is, the covering associated to the kernel of the holonomy representation of the leaf.

Definition 5.4. Let $\pi : \widehat{L}_x \rightarrow L_x$ denotes the holonomy covering of the leaf L_x through $x \in X$. A pointed pattern (P, p) is around $x \in X$ if there is a copy of the pattern $P \subset \widehat{L}_x$ such that $\pi(p) = x$. Call

$$X_{(P,p)} = \{ x \in X \mid (P, p) \text{ is around } x \}.$$

to the set of all those points.

Lemma 5.5. *The set X_P is clopen.*

Proof. Fix $x \in X_P$. For each tile Δ of P we can fix a path of tiles $\{\Delta_0, \Delta_1, \dots, \Delta_k\}$, where Δ_0 is the tile containing p and $\Delta_k = \Delta$. The projection of the path defines a path of plaques on M , and the associated holonomy map is given by

$$\gamma_{p,\Delta} = \sigma_{\Delta_{k-1}, \Delta_k} \circ \dots \circ \sigma_{\Delta_1, \Delta_2} \circ \sigma_{\Delta_0, \Delta_1}.$$

The map above does not depend on the path itself but on Δ as the different paths are homotopic relatively to its ends in the holonomy covering. It is obvious that

$$X_P = \bigcap_{\Delta \in P} \text{dom } \gamma_{p,\Delta}$$

By definition of the holonomy transformations $\sigma_{\Delta, \Delta'}$, their domains are clopen so $\text{dom } \gamma_{p,\Delta}$ are also clopen. Finally, there are a finite number of simplexes, hence X is clopen. \square

Now, given a pattern P there is a natural map $P \times X_P \rightarrow M$ given as the gluing of the embeddings $\Delta \times X_P \subset \Delta \times X_\Delta \rightarrow M$. Those maps shows the following result:

Lemma 5.6. *Given a pointed pattern (P, p) , there is a singular box*

$$\psi_P : P \times X_P \rightarrow M,$$

that is, a foliated map such that

- (1) *is a homeomorphism restricted to each transversal $\{*\} \times X_P$ and*
- (2) *is a local homeomorphism restricted to each plaque $P \times \{*\}$.*

Definition 5.7. A tile decoration of \mathcal{K} is a disjoint clopen cover of $X_{\mathcal{K}}$.

It is clear the associated set X_P for a decorated pattern is also clopen, as if is enough to restrict the domains and images of the holonomy transformations to the clopen sets of the decoration. Hence, Lemma 5.6 also holds for decorated patterns.

5.3. Codimension zero laminations are inverse limits. The aim of this section is to prove the Theorem:

Theorem 5.8. *Any codimension zero lamination (M, \mathcal{L}) is homeomorphic to an inverse limit $\varprojlim (S_k, f_k)$ of branched manifolds S_k and submersions $f_k : S_k \rightarrow S_{k-1}$.*

Proof. For now on let \mathcal{K} denote a simplicial box decomposition as the one of Theorem 5.2. The axis $X_{\mathcal{K}}$ of \mathcal{K} is a totally disconnected compact metrizable space, therefore we can choose a countable basis \mathfrak{B} of the topology of $X_{\mathcal{K}}$ such that

- (1) each set $B \in \mathfrak{B}$ is clopen and;

- (2) the basis is written as a disjoint union of *floors* $\mathfrak{B} = \bigsqcup \mathfrak{B}_k$ such that
- (a) for each $B \in \mathfrak{B}_{k+1}$ there exists $B' \in \mathfrak{B}_k$ such that $B \subsetneq B'$ and;
 - (b) \mathfrak{B}_k is an open cover of $X_{\mathcal{K}}$.

For each $p \in X_{\mathcal{K}}$, let \tilde{L}_p be the holonomy covering of L_p the leaf through p . Fix a lift \tilde{p} of p . Let Δ_p denote the simplex containing p , and $\Delta_{\tilde{p}}$ the lift of it to \tilde{L}_p . In other words, $\Delta_{\tilde{p}}$ is the simplex whose barycenter is \tilde{p} .

The first branched manifold S_1 . Fix the decoration of tiles $\mathfrak{D}_1 = \mathfrak{B}_1$. From now until next step all patterns will be decorated by \mathfrak{D}_1 . Consider the pattern $P_p^1 = \star(\Delta_{\tilde{p}})$. It is clear P_p^1 does not depend on the election of \tilde{p} , as deck transformations relate all possible elections. Associated to P_p^1 there is a clopen set $X_{P_p^1}$. By definition it is obvious that two sets $X_{P_p^1}$ and $X_{P_{p'}^1}$ agree or are disjoint. As $X_{\mathcal{K}}$ is compact, there are finitely many of them, or equivalently, there are finitely many decorated patterns $\{P_1^1, P_2^1, \dots, P_{\ell_1}^1\}$. We have then the associated sets $X_{P_i^1}$ and singular boxes $\varphi_i^1 : P_i \times X_{P_i^1} \rightarrow M$. We define an equivalence relation \sim_1 on M generated by

$$x \sim_1 y \iff \exists \varphi_i^1 \text{ s.t. } pr_1 \circ \varphi_i^{1-1}(x) = pr_1 \circ \varphi_i^{1-1}(y).$$

That relations is just the collapsing the transversals of the singular boxes on M . Set $S_1 = M/\sim_1$ and q_1 the quotient map.

The space S_1 has a natural branched manifold structure. It is obvious S_1 is covered by the images of the starts of points $\star x = \bigcup_{\Delta \ni x} \Delta$, with $x \in M$. Moreover, by the compactness of S_1 (there are just finitely many patterns on M) finitely many of them are enough, and can be chosen to be vertices of the triangulation. Sadly, those disks might not map to disks on S_1 , as some of their simplexes may meet in other ways in M , so the quotient is not a disk (see Figure 2a). Define the disk D_x as

$$D_x = \{ y \in \star x \mid \text{bar}_x(y) \geq 2/3 \},$$

where $\text{bar}_x(y)$ denotes the barycentric coordinate of y associated to x on the corresponding simplex (see Figure 2a). Those disks are embedded as disks on S_1 . Now, given $s \in S_1$ the image of a vertex, define the open set

$$U_s = \bigcup_{q_1(x \in M) = s} q_1(D_x).$$

It is straightforward to see the U_s open sets with the disks D_x form a branched manifold atlas for S_1 .

The branched manifolds S_k . Now, by induction suppose we have the partition by clopen sets $\{X_{P_i^{k-1}}\}_{i=1}^{\ell_{k-1}}$ and S_{k-1} . We can define the decoration

$$\mathfrak{D}_k = \{ X_{P_i^{k-1}} \cap B \mid 1 \leq i \leq \ell_{k-1} \text{ and } B \in \mathfrak{B}_k \}.$$

So now, for each $p \in P$, consider the pattern decorated by \mathfrak{D}_k

$$P_p^k = \star P_p^{k-1} = \star \cdots \star \Delta_{\tilde{p}},$$

using the same notation above. Proceeding as before, we obtain finite partition by clopen sets $\{X_{P_i^k}\}_{i=1}^{\ell_k}$ and the branched manifold S_n and the quotient map q_k .

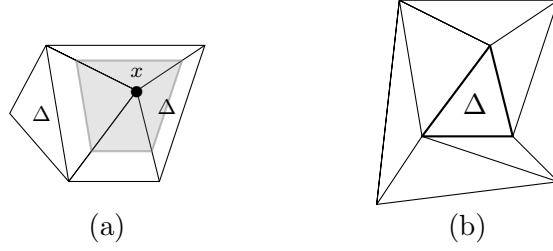


FIGURE 2. (a) The set $\star x$ and the disk D_x (in grey). (b) The pattern $\star \Delta$: all tiles meeting Δ .

The branched manifold system. By construction given P_i^k there exists P_j^{k+1} such that $X_{P_j^{k+1}} \subset X_{P_i^k}$. It is clear, P_j^{k+1} is just the star of P_i^k with a finer decoration. Even more, the natural maps embedding the associated boxes agree. This implies there is a natural map $f_{k+1} : S_{k+1} \rightarrow S_k$ such that

$$\begin{array}{c}
 M \xrightarrow{q_1} S_1 \\
 \quad \searrow q_2 \quad \searrow q_3 \\
 \quad \quad \quad \cdots \quad S_3 \xrightarrow{f_3} S_2 \xrightarrow{f_2} S_1
 \end{array}$$

is commutative. Therefore, there is a continuous map $q_\infty : M \rightarrow S_\infty$.

Finishing the proof. The map q_∞ is a homeomorphism. As each quotient map q_n is onto, the map q_∞ has dense image. But M is compact and S_∞ is Hausdorff, therefore q_∞ is onto. On the other hand, q_∞ is into. Fix x and $y \in M$ such that $q_\infty(x) = q_\infty(y)$. This implies those points share decorated patterns of any size. Therefore $x = y$. As q_∞ is continuous bijection from a compactum to a Hausdorff space, it is a homeomorphism. \square

6. INVERSE LIMITS OF REGULAR COVER MAPS

Adding conditions to the inverse limit structure, we can get interesting results about the transverse dynamics of the corresponding laminations. One interesting case is the equicontinuous foliated spaces.

Consider a projective system

$$(4) \quad S_1 \xleftarrow{f_2} S_2 \xleftarrow{f_3} S_3 \xleftarrow{f_4} \cdots$$

where S_k are compact finitely dimensional manifold and f_k are regular covering maps. To avoid trivial situations, assume all bonding maps are not homeomorphisms, that is, the degree of each f_k is greater than 1. The inverse limit $S_\infty = \varprojlim (S_k, f_k)$ has a natural structure of matchbox manifold as it is obviously flattening, even on those trivial cases.

Theorem 6.1. *Each standard transversal of S_∞ has a natural structure of profinite group. Moreover, those structures are all isomorphic.*

Proof. Fix a thread $(x_k) \in S_\infty$. For now on x_k will be the base point of S_k for all homotopy and cover related tools.

Consider the transversal $T_1 = q_1^{-1}(x_1)$. Now if $k > k'$,

$$f_{k,k'} = f_k \circ f_{k-1} \circ \cdots \circ f_{k'+1} : S_k \rightarrow S_{k'}$$

is also a regular covering map. By definition $f_{k,k-1} = f_k$. Denote by $\Delta_{k,k'}$ the deck transformation group of $f_{k,k'}$ acting on $D_{k,k'} = f_{k,k'}^{-1}(x_{k'})$. As $f_{k,k'}$ is regular, $\Delta_{k,k'}$ acts freely and transitively on $D_{k,k'}$.

We have then the identification

$$(5) \quad \delta \in \Delta_{k,1} \mapsto \delta(x_k) \in D_{k,1}.$$

Therefore the transversal T_1 can be thought as

$$T_1 = q_1^{-1}(x_1) = \varprojlim (D_{k,1}, f_{k,k-1}|_{D_{k,1}}) \cong \varprojlim (\Delta_{k,1}, f_{k,k-1}|_{\Delta_{k,1}})$$

with the previous identification. Let us see $f_{k,k-1}$ is a homomorphism of groups considered with domain $\Delta_{k,1}$ and range $\Delta_{k-1,1}$.

As $f_{k,1}$ is a regular covering $\pi_1(S_k) \trianglelefteq \pi_1(S_1)$ and $\Delta_{k,1} = \pi_1(S_1)/\pi_1(S_k)$ (with the usual identifications). As the same holds with $f_{k,k'}$ we can think that

$$\cdots \trianglelefteq \pi_1(S_3) \trianglelefteq \pi_1(S_2) \trianglelefteq \pi_1(S_1).$$

By the isomorphism theorem we have the identification

$$\Delta_{k,1} = \frac{\Delta_{k+1,1}}{\Delta_{k+1,k}}.$$

Remember that $f_{k,k-1}$ is the quotient map $S_k \rightarrow S_k/\Delta_{k,k-1} = S_{k-1}$. This, with the identification (5) and the normality of the groups

$$(6) \quad \begin{aligned} f_{k,k-1}(g) &\stackrel{(5)}{=} f_{k,k-1}(g(x_k)) = \{ h(g(x_k)) \mid h \in \Delta_{k,k-1} \} \\ &= \{ g(\tilde{h}(x_k)) \mid \tilde{h} \in \Delta_{k,k-1} \} = g \circ \bar{h}(\{ \tilde{h}(x_k) \mid \tilde{h} \in \Delta_{k,k-1} \}) \end{aligned}$$

for any $\bar{h} \in \Delta_{k,k-1}$. Again as $f_{k,k-1}$ is a quotient map

$$f_{k,k-1}(x_k) = x_{k-1} = \{ \tilde{h}(x_k) \mid \tilde{h} \in \Delta_{k,k-1} \},$$

so we conclude from (6)

$$f_{k,k-1}(g) = g\Delta_{k,k-1}.$$

Therefore the transversal T_1 has the structure of the profinite group

$$\varprojlim (\Delta_{k,1}, f_{k,k-1}).$$

Notice that the identity of the group is identified with (x_k) . But the election of another thread does not change the isomorphism class.

Finally, it is well known that the isomorphism class does not change by telescoping, so for any transversal $q_k^{-1}(\tilde{x}_k)$ for any $\tilde{x}_k \in S_k$ we obtain the same profinite group structure. \square

All leaves of S_∞ share the fundamental group $\bigcap_k \pi_1(S_k)$ (see [13]). Therefore each leaf L is a normal covering space of S_1 having

$$\Delta_{\infty,1} = \frac{\pi_1(S_1)}{\bigcap_k \pi_1(S_k)}$$

as deck transformation group. The quotient map $\Delta_{\infty,1} \rightarrow \Delta_{k,1}$ gives rise to a natural representation of $\Delta_{\infty,1}$ in T_1 :

$$g \in \Delta_{\infty,1} \mapsto (g\Delta_{k,1}) \in \varprojlim (\Delta_{k,1}, f_{k+1,k}).$$

This representation is faithful as if $g \in \Delta_{\infty,1}$ such that $g\Delta_{k,1} = \Delta_{k,1}$ for all k implies g acts trivially in all levels S_k . Then g acts trivially on L , so it should be the identity.

In that way, $\Delta_{\infty,1}$ is a subgroup of T_1 . Hence we have the free action of $\Delta_{\infty,1}$ on T_1 by left translations. We have then a diagonal action given by

$$\Psi : h \cdot (\lambda, (\tilde{x}_k)_k) \mapsto (h^{-1}\lambda, h(x_k)_k).$$

It is straightforward to see that:

Lemma 6.2. *The lamination S_∞ is the suspension of Ψ . Therefore the transverse dynamics of S_∞ given by the action of $\Delta_{\infty,1}$ on T_1 . \square*

As T_1 is a group we can consider a left invariant distance, in fact, we can construct it explicitly. Consider the discrete distance $\delta_{\Delta_{k,1}}$ on each group $\Delta_{k,1}$ which is 1 on different elements. The product metric restricted to T_1

$$(7) \quad d((x_k), (y_k)) = \sum_{k=2}^{\infty} \frac{1}{2^k} \delta_{\Delta_{k,1}}(x_k, y_k)$$

is invariant by the action of T_1 on itself, so it is invariant by the action of the subgroup $\Delta_{\infty,1}$. Therefore any lamination given as an inverse limit of manifolds and regular coverings preserves a distance function. A. Clark and S. Hurder shows in [7] that those laminations are exactly the equicontinuous ones.

Theorem 6.3. *A equicontinuous lamination of codimension zero preserves a transverse metric.*

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